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Some Formulas Calculate the Volume and Height of Tetrahedra and Some Applications of Tetrahedra in Engineering

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Abstract **- A tetrahedron is a pyramid with four sides that are equilateral triangles. It is one of the basic building blocks of an object. So it is widely used in engineering. Determining the volume of tetrahedra is very important. It helps us to determine the exact mass of an object. This is very important work in engineering design. In addition to the tetrahedral volume formulas, this article gives some more tetrahedral volume formulas and some applications of its in engineering.**

Keywords: Tetrahedron, Tet Traingle, Trai Equation, Eqt.

I. INTRODUCTION

The attractiveness of technology to simple structures shown in the aestheticization process is increasing. Simple geometries are considered to be the best response to mass production modes. The study and application of polyhedra to create architectural spatial forms is an inevitable trend. And the trend in the design of modern architectural works is how to create large space beyond the span, simple but iconic architectural shapes. Therefore, the application of equal and semi-uniform polyhedra into the design and construction of architectural works in the world is very necessary and deserves attention. Not confined to lines, meeting the architectural creativity of the design engineers, being able to structure complex shapes without columns is one of the many advantages of the space roof trusses appears more and more in the diverse architectural complex in the world today.

II. GIVEN SOME FORMULAS TO CALCULATE THE VOLUME OF A TRAINGULAR PYRAMID

2.1 Irregular tetrahedra

ABCD is a triangular pyramid. It has edges: a, b, c. Three angles at vertex D are: ADB, BDC and ADC (see fig 1)

V is is the Volume of the regular tetrahedron

B is the area of the base

H is heigt

We have a formular:

Volume of the regular tetrahedron

= 1/3 × area of the base × height

$$V = \frac{1}{3} BH \quad (1)$$

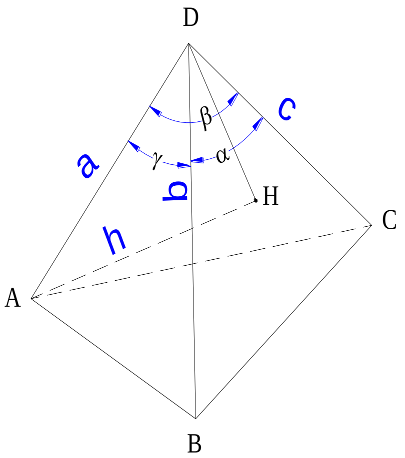

Figure 1

Set:

$\widehat{BDC} = \propto$

$\widehat{ADC} = \beta$

$\widehat{BDA} = \gamma$

$\overrightarrow{DA} = a, \overrightarrow{DB} = b, \overrightarrow{DC} = c$

$S_{DBC} = \frac{1}{2} bc \sin\alpha, AH \perp (DBC)$

$Set: h = AH$

$$V_{DABC} = \frac{1}{6} bch \sin\alpha \quad (2)$$

We have:

$$\overrightarrow{DH} = x.\overrightarrow{DB} + y.\overrightarrow{DC} = x.\vec{b} + y.\vec{c} \quad (3)$$

$$\overrightarrow{AH} = \overrightarrow{DH} - \overrightarrow{DA} = x.\vec{b} + y.\vec{c} - \vec{a} \quad (4)$$

We solve the system of equations in perpendicular conditions:

$$\begin{Bmatrix} \overrightarrow{AH}.\overrightarrow{DB} = 0 \\ \overrightarrow{AH}.\overrightarrow{DC} = 0 \end{Bmatrix} (5) \quad \rightarrow \begin{Bmatrix} (x.\vec{b} + y.\vec{c} - \vec{a})\vec{b} = 0 \\ (x.\vec{b} + y.\vec{c} - \vec{a})\vec{c} = 0 \end{Bmatrix} (6)$$

$$\rightarrow \begin{Bmatrix} xb^2 + ybc\cos\alpha - ab\cos\gamma = 0 \quad (10) \\ xb\cos\alpha + yc^2 - ac\cos\beta = 0 \; (11) \end{Bmatrix} \quad (7)$$

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$$\begin{cases} D = \begin{vmatrix} b^2 & bc\cos\alpha \\ bc\cos\alpha & c^2 \end{vmatrix} = b^2c^2 - b^2c^2cos^2\alpha = b^2c^2sin^2\alpha \\ Dx = \begin{vmatrix} ab\cos\gamma & bc\cos\alpha \\ ab\cos\beta & c^2 \end{vmatrix} = abc^2(\cos\gamma - \cos\alpha\cos\beta) \\ Dy = \begin{vmatrix} b^2 & ab\cos\gamma \\ bc\cos\alpha & ac\cos\beta \end{vmatrix} = ab^2c(\cos\beta - \cos\alpha\cos\gamma) \end{cases} \quad (8)$$

$$\rightarrow \begin{cases} x = \frac{D}{Dx} = \frac{a(\cos\gamma - \cos\alpha\cos\beta)}{bsin^2\alpha} \\ y = \frac{D}{Dy} = \frac{a(\cos\beta - \cos\alpha\cos\gamma)}{csin^2\alpha} \end{cases} \quad (9)$$

So:

$h^2 = \left(x.\vec{b} + y.\vec{c} - \vec{a}\right)^2 =$

$a^2 + x^2b^2 + y^2c^2 + 2xybc\cos\beta - 2yac\cos\beta - 2xab\cos\gamma$ (12)

$= x.(10) + y.(11) + a^2 - abx\cos\gamma - acy\cos\beta$

$= a^2 - abx\cos\gamma - acy\cos\beta$

$\rightarrow h^2sin^2\alpha = a^2sin^2\alpha - abxsin^2\alpha\cos\gamma - acysin^2\alpha\cos\beta$

$= a^2sin^2\alpha - a^2\cos\gamma(\cos\gamma - \cos\alpha\cos\beta) - a^2\cos\beta(\cos\beta - \cos\alpha\cos\gamma)$

$= a^2(1 + 2\cos\alpha\cos\beta\cos\gamma - cos^2\alpha - cos^2\beta - cos^2\gamma)$

$\rightarrow h\sin\alpha$

$= a\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - cos^2\alpha - cos^2\beta - cos^2\gamma}$ (13)

$V_{DABC} = \frac{1}{3}\frac{1}{2}bca\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - cos^2\alpha - cos^2\beta - cos^2\gamma}$ (14)

$V_{DABC} = \frac{1}{6}abc\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - cos^2\alpha - cos^2\beta - cos^2\gamma}$ (15)

$V_{DABC} = \frac{abc}{6}\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - cos^2\alpha - cos^2\beta - cos^2\gamma}$ (16)

If: $\alpha + \beta + \gamma = 180°$

We have:

$cos^2\alpha + cos^2\beta + cos^2\gamma = 1 - 2\cos\alpha\cos\beta\cos\gamma$ (17)

Set $Vt = cos^2\alpha + cos^2\beta + cos^2\gamma$

$Vp = 1 - 2\cos\alpha\cos\beta\cos\gamma$

We prove this result: $Vt = Vp$

We have:

$$cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$$

$$cos^2\beta = \frac{1 + \cos 2\beta}{2}$$

$$cos^2\gamma = \frac{1 + \cos 2\gamma}{2} = \frac{1 + \cos 2(\alpha + \beta)}{2}$$

Hence:

$$Vp = \frac{3}{2} + \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\alpha\cos 2\beta - \sin 2\alpha\sin 2\beta) \quad (18)$$

$Vt = 1 + 2\cos\alpha\cos\beta\cos(180° - \alpha - \beta)$

$= 1 + 2\cos\alpha\cos\beta\cos(\alpha + \beta)$

$= 1 + 2\cos\alpha\cos\beta(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$= 1 + 2cos^2\alpha cos^2\beta - 2\sin\alpha\cos\alpha\sin\beta\cos\beta$

$= (1 + \cos 2\alpha)(1 + \cos 2\beta) - \frac{1}{2}\sin 2\alpha\sin 2\beta$

$$= \frac{3}{2} + \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\alpha\cos 2\beta - \sin 2\alpha\sin 2\beta) \quad (19)$$

See (18) and (19) we have: $Vp = Vt$

Therefore:

$V = \frac{abc}{3}\sqrt{\cos\alpha\cos\beta\cos\gamma}$ (20)

$V = \frac{1}{3}\sqrt{\left(\vec{a}.\vec{b}\right)\left(\vec{b}.\vec{c}\right)\left(\vec{c}.\vec{a}\right)}$ (21)

We also have another way to calculate the volume of tetrahedra by using Simsom's theorem.

Take 3 points M, N, P on the lines DA, DB, DC respectively such that: DM = DN = DP = 1 (Length unit). (See fig 2).

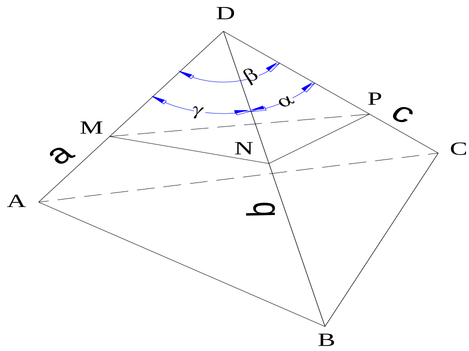

Figure 2

We will determine the volume of the tetrahedron DMNP (see fig 3)

From that deduce volume of tetrahedron ABCD

V_{MNPQ} is the volume of the tetrahedron MNPQ

V_{ABCD} is the volume of the tetrahedron MNPQ

Based on the Simmom theorem we have:

$V_{MNPQ} = V_{ABCD}.k$

$$k = \frac{DM}{DA}\frac{DN}{DB}\frac{DP}{DC} = \frac{1}{DA.DB.DC} = \frac{1}{abc}$$

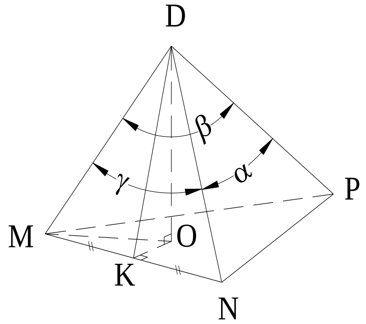

Figure 3

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$V_{DMNP} = \frac{DO.S_{MNP}}{3}$ (See Fig 3)

S_{MNP} is the area of the triangle MNP

DO is the height of the tetrahedron DMNP

O is the center of the circumcircle of the triangle MNP

K is midle point of MN (KM=KN)

DM=DN→KM⊥MN→DK=DM.cos$\frac{\gamma}{2}$=cos$\frac{\gamma}{2}$

MN=2KM=2DM.sin$\frac{\gamma}{2}$ =2sin$\frac{\gamma}{2}$

Do the same we also have:

MP=2sin$\frac{\beta}{2}$

NP=2sin$\frac{\alpha}{2}$

OM=$R_c$is the radius of the circumcircle of the triangle MNP

SetCv is the half-perimeter of the triangle MNP

$Cv = \left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)$ (22)

$S_{MNP} = \frac{MN.NP.PN}{4R_c} = \frac{1}{4R_c} 2\sin\frac{\alpha}{2} 2\sin\frac{\beta}{2} 2\sin\frac{\gamma}{2} = \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{R_c}$

(23)

$Rc = \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{S_{MNP}}$ (24)

According to Heron formula we have:

$S_{MNP} = \sqrt{Cv\left(Cv - 4\sin\frac{\alpha}{2}\right)\left(Cv - 4\sin\frac{\beta}{2}\right)\left(Cv - 4\sin\frac{\gamma}{2}\right)}$

(25)

$$Rc = \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sqrt{Cv\left(Cv - 4\sin\frac{\alpha}{2}\right)\left(Cv - 4\sin\frac{\beta}{2}\right)\left(Cv - 4\sin\frac{\gamma}{2}\right)}}$$ (26)

Set $S_{MNP} = u$

$Rc = \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{u}$ (27)

$DO = \sqrt{DM^2 - OM^2} = \sqrt{DM^2 - Rc^2}$ (28)

$DO = \sqrt{1 - \frac{\left(2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)^2}{u^2}}$ (29)

$V_{DMNP} = \frac{1}{3} DO.S_{MNP}$ (30)

$V_{DMNP} = \frac{\sqrt{u^2 - \left(2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)^2}}{3}$ (31)

$V_{DABC} = V_{DMNP}.k = \frac{k}{3}\sqrt{u^2 - \left(2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)^2}$

$= \frac{k}{3}\sqrt{\left(u - 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)\left(u + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)}$

(32)

2.2 The nearly equilateral tetrahedra

If DA=BC=a, DB=AC=b and DC=AB=c then this tetrahedron will have four faces surrounded by equal triangles. This tetrahedron has 4 faces that are all equal triangles. So the four heights of the tetrahedra are also equal. See fig 4.

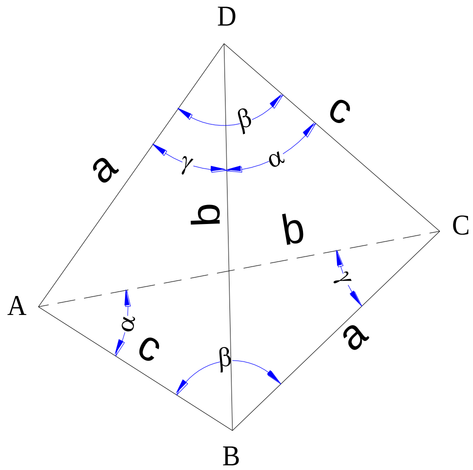

Figure 4

These are nearly equilateral tetrahedra, then the volume of it is:

$V = \frac{abc}{3}\sqrt{\cos\alpha\cos\beta\cos\gamma}$ (33)

We have:

$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$ (34)

$\cos\beta = \frac{a^2+c^2-b^2}{2ac}$(35)

$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$(36)

Hence we have:

$V = \frac{abc}{3}\sqrt{\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}{8a^2b^2c^2}}$(37)

$V = \frac{\sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}}{6\sqrt{2}}$ (38)

$H = \frac{3V}{B} = \frac{2.3abc}{3ab\sin\gamma}\sqrt{\cos\alpha\cos\beta\cos\gamma}$

$= \frac{2c}{\sin\gamma}\sqrt{\cos\alpha\cos\beta\cos\gamma}$ (39)

$H = \frac{2a}{\sin\beta}\sqrt{\cos\alpha\cos\beta\cos\gamma}$ (40)

$H = \frac{2b}{\sin\alpha}\sqrt{\cos\alpha\cos\beta\cos\gamma}$ (41)

We have:

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$$\sin\alpha = \sqrt{1 - cos^2\alpha} \quad (42)$$

$$\sin\alpha = \sqrt{1 - \frac{(b^2+c^2-a^2)^2}{4b^2c^2}} \quad (43)$$

$$\sin\alpha = \frac{1}{2bc}\sqrt{(2bc)^2 - (b^2+c^2-a^2)^2} \quad (44)$$

$$\sin\alpha = \frac{1}{2bc}\sqrt{(a+b-c)(a+c-b)(b+c-a)(a+b+c)} \quad (45)$$

$$H = \frac{4b^2c\sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}}{2\sqrt{2}abc\sqrt{(a+b-c)(a+c-b)(b+c-a)(a+b+c)}} \quad (46)$$

$$H = \frac{b\sqrt{2(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}}{a\sqrt{(a+b-c)(a+c-b)(b+c-a)(a+b+c)}} \quad (47)$$

III. SOME APPLICATIONS OF TETRAHEDRA IN ENGINEERING

3.1 Tetrahedra with a twist (see fig 5)

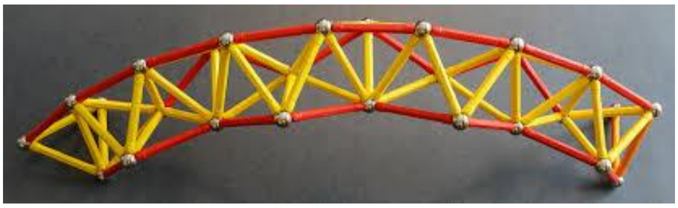

Figure 5

The tetrahelix of Fuller's Synergetics consists of face bond regular tetrahedra. The mathematics for this spiraling structure is quit interesting.

3.2 Applications of tetrahedra in architecture (See fig 6, fig 7)

Figure 6

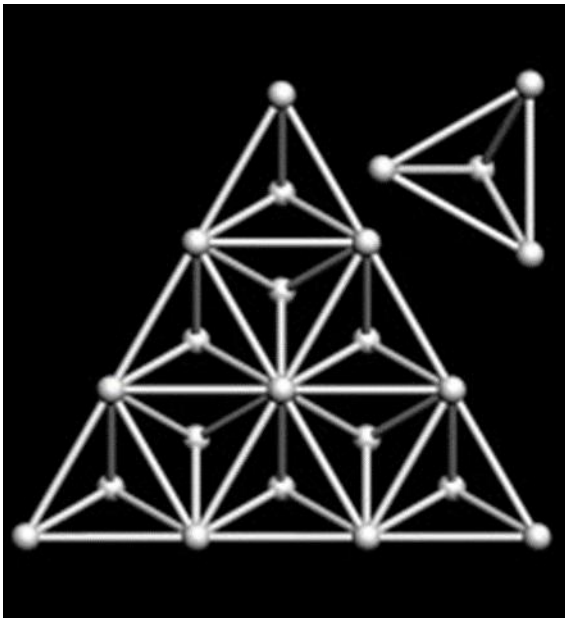

Figure 7

3.3 The vertices of the regular tetrahedra lay on three cylindrical helixs of the surface of a cylinder. (See fig 8)

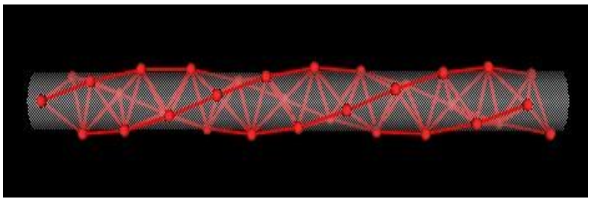

Figure 8

3.4 Variational tetrahedral meshing (See fig 9)

Application of finite element method

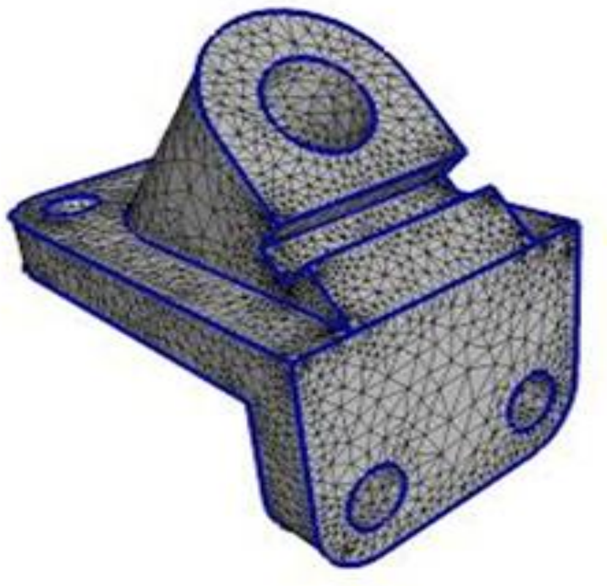

Figure 9

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3.5 The Stella octangula. (See fig 10)

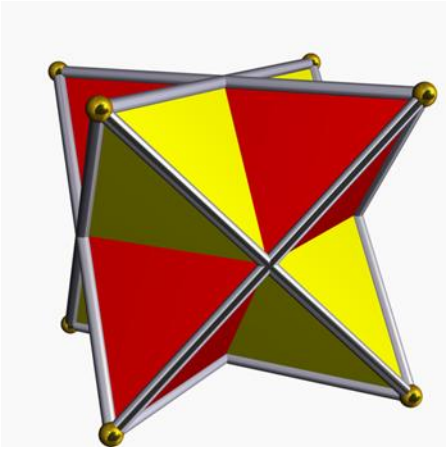

Figure 10

3.6 Create chamfer surfaces in engineering. (See fig 10)

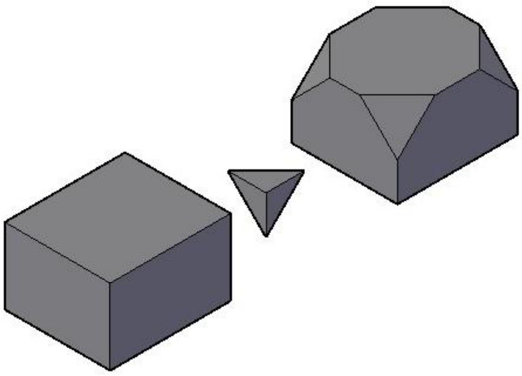

Figure 11

IV. CONCLUSION

In short, a tetrahedron is a shape that has quite a wide range of applications in practice. It is used in domed structures, canopy in supermarkets and large commercial centers. In addition it is also used to represent molecular structure in chemistry. In machine manufacturing, chamfer faces of machine parts are created. The tetrahedron is also used in the finite element method. The cradle of the object is divided in the form of a triangular grid. Therefore, Objects considered to be composed of the smallest elements are tetrahedra.

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